MATH - 21260 RECITATION NOTES

PRANAY GUNDAM

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Contents

Recitation 1.

1.1 Course Information

Communication and office hours. The best place to contact me is via email, pgundam@andrew.cmu.edu. My office hours are - for now - on zoom from 7:00pm to 9:00 pm on Wednesdays. You can find the zoom link on canvas.

Homework and Quizzes. HW is divided into two parts, a written and an online section. Late written HW will be accepted with a deduction (the specifics of which can be found on the syllabus) and must be submitted onto gradescope. More specifics about the online WebWork HW can be found on canvas. We will sometimes have Chapter Tests in class, more information to come.

Changes to this document. If you would like any of the answers to any of the exercises in this document or notice any error, feel free to email me at pgundam@andrew.cmu.edu.

1.2 Differential Equation Classification

Before we delve into methods of solving differential equations we should look to a couple of definitions and ways that we can classify them. This is important because after classification, the method we need to use to solve the DE becomes very clear. One note is that the following definitions may not be thoroughly rigorous, they are meant to be in a format that is readable and easy to learn rather than adhering to proper notation very strictly.

Definition 1.1. A <u>Linear Differential Equation</u> is any differential equation of the form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_0(x)y = g(x)$$

Where each $a_n(x), a_{n-1}(x), \ldots, a_0(x), g(x)$ is a function of x. Any DE that cannot be put into this form may be considered a non-linear differential equation.

Definition 1.2. The <u>order</u> of a differential equation is the differential coefficient of the highest derivative in a DE. For example, the order of the equation $y^{(n+1)} + y^{(n)} + y^{(4)} = 0$ is $n+1$.

Definition 1.3. <u>Autonomous</u> differential equations are equations of the form $\frac{dy}{dx} = h(y)$ for some function $h(y)$.

Definition 1.4. <u>Partial differential equations</u> are equations that include partial derivatives. We do not focus on solving these as much in this class because they require more sophisticated tools

Definition 1.5. <u>Separable differential equations</u> are equations of the form $\frac{dy}{dx} = h(y)g(x)$ where $h(y)$ and $g(x)$ are some functions of y and x respectively such that we can "separate" the x components from the y components.

1.3 Separable Differential Equations

The informal method to solve separable equations is to move all the x terms onto one side and the y terms on to the other side and then take an implicit integral and solve for any remaining initial conditions. To generalize:

$$\frac{dy}{dx} = h(y)g(x) \tag{1}$$

$$\frac{dy}{h(y)} = g(x) \cdot dx \tag{2}$$

$$\int \frac{dy}{h(y)} = \int g(x) \cdot dx + c \tag{3}$$

After which we can solve explicitly for an expression for y if we desire. Let's discuss a couple more examples in depth.

Example 1.6.

$$\frac{dy}{dx} = e^{3x-5y}$$

Solution.

$$\begin{aligned}\frac{dy}{dx} &= e^{3x-5y}\\ &= e^{3x} \cdot e^{-5y}\end{aligned}$$

Thus we can then write

$$e^{5y} \cdot dy = e^{3x} \cdot dx \Longrightarrow \int e^{5y} \cdot dy = \int e^{3x} \cdot dx \Longrightarrow \frac{1}{5}e^{5y} = \frac{1}{3}e^{3x} + c$$

And finally solving explicitly for y we have

$$5y = \log\left(\frac{5}{3}e^{3x} + c\right) \Longrightarrow \boldsymbol{y = \frac{1}{5}\log\left(\frac{5}{3}e^{3x} + c\right)}$$

Example 1.7.

$$x^2\frac{dy}{dx} = y - xy, \quad y(-1) = 1$$

Solution.

$$\begin{aligned}x^2\frac{dy}{dx} &= y - xy\\ \frac{dy}{dx} &= \frac{y(1-x)}{x^2}\end{aligned}$$

Thus we can then write

$$\frac{1}{y} \cdot dy = \frac{1-x}{x^2} \cdot dx \Longrightarrow \int \frac{1}{y} \cdot dy = \int \frac{1-x}{x^2} \cdot dx \Longrightarrow \ln|y| = -\frac{1}{x} - \ln|x| + c$$

Plugging in our initial condition we have

$$\ln|1| = 1 - \ln|1| + c \Longrightarrow c = -1$$

Finally, solving explicitly we have

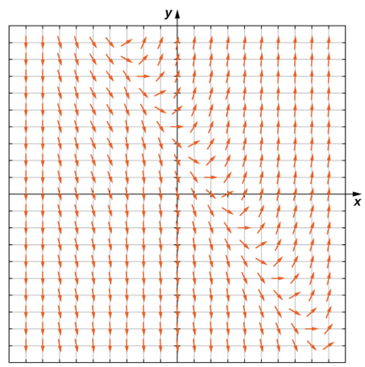
$$\boldsymbol{y = \frac{e^{-(1+1/x)}}{x}}$$

Recitation 2.

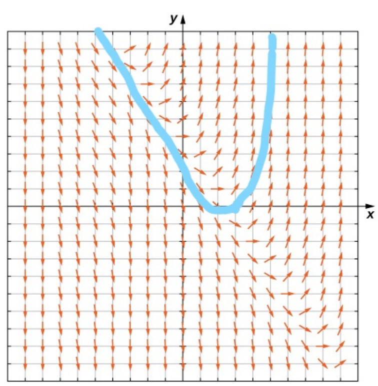
2.4 Graphing Solutions

Sometimes, in addition to - or instead of - a numerical solution, it is helpful to look at a more graphical representation of potential solutions for a given differential equation. For first order DEs we use slope fields to help visualize all the potential solutions on the $\mathbb{R}^2$ plane. To formalize:

Definition 2.8. Direction fields/Slope Fields are a graphical representation of the slope given by a differential equation at points on a graph. So given some $\frac{dy}{dx} = f(x, y)$ we simply plug in some x, y and indicate the slope of a potential solution curve at that point. Remember that the interpretation of $\frac{dy}{dx}$ is just the slope. For example the slope field below is one that corresponds to the D.E. $y' = 3x + 2y - 4$:

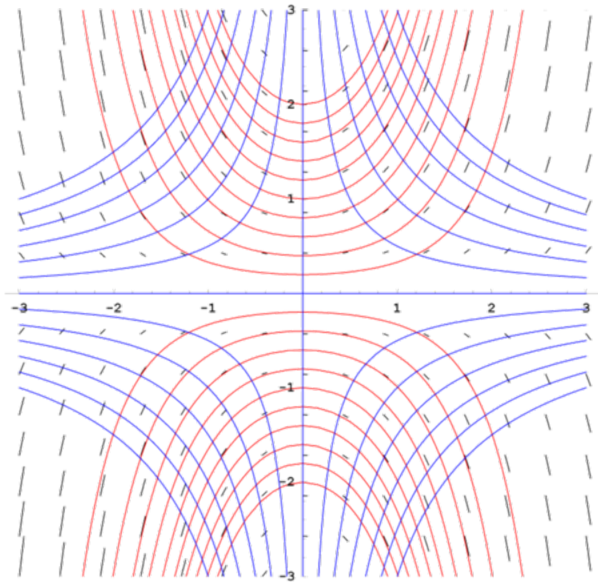


The important takeaway is that we can draw different solutions curves (the exact one of which depends on our initial condition) by drawing lines that connect the slopes at each point. In the example above, if we wanted to find the solution for $y' = 3x + 2y - 4$ with initial condition $y(4) = 0$ we would start at the point $(4, 0)$ and connect the points on the graph using the slopes lines at each point as a guide. Below is a *very very rough* approximation of what a potential solution would look like.



Definition 2.9. We define **Isoclines** as the resulting equations that come by setting $f(x, y) = c$ for some

$c \in \mathbb{R}$ where $y' = f(x, y)$. To visualize, below is a slope field of the equation $y' = xy$ with potential solutions curves in red and isoclines in blue.



The purpose of isoclines is that along an isocline curve, each point has the same slope and so this makes it easier to draw in our slope fields.

2.5 Mathematic Models

Differential equations can be used to model certain scenarios and give us information that we wouldn't be able to discern without using differential equations to model. The consistent theme for scenarios that are modeled using differential equations is that quantities tend to have a variable rate of change.

We shall discuss a couple of models but as a basic format solving problems that use mathematic models involves three steps:

1. Identify the model to use.
2. Determine the value of the constants in the model.
3. Solve the differential equation.

Often times steps 2 and 3 are swapped, sometimes the question asks us to find the final model and other times the question asks us to find the value of the constants.

Theorem 2.10. Population Dynamics: We denote the rate at which populations change as

$$\frac{dP}{dt} = kP$$

where $P(t)$ is the size of the population at time t and k is the constant of proportionality.

Theorem 2.11. Radioactive Decay: This model is very similar to population dynamics where $A(t)$ of the substance remaining at time t and k is the constant of proportionality where $k < 0$ if this is decay and $k > 0$ is this is growth.

Theorem 2.12. Mixtures: If $A(t)$ denotes the amount of salt (measured in pounds) in the tank at time t, then the rate at which $A(t)$ changes is a net rate:

$$\frac{dA}{dt} = (\text{input rate of salt}) - (\text{output rate of salt})$$

This point in the course we will not be doing any in-depth examples but be on the lookout for them in the future.

Recitation 3.

3.6 First Order Linear ODEs

To begin, we classify **First order linear ODEs** as a DE that can be written in the form $\frac{dy}{dx} + p(x)y = g(x)$.

I will spare the machinery of the proof and get straight to the method of solving First Order Linear DEs. To generalize, for some differential equation

$$y' + P(x)y = Q(x)$$

we have the solution

$$y = \frac{\int \mu \cdot Q(x)\, dx + c}{\mu}$$

where

$$\mu = e^{\int P(x)\, dx}$$

Example 3.13. Solve

$$ty' + 2y = t^2 - t + 1, \quad y(1) = \frac{1}{2}$$

Solution. Let's first rewrite our expression into a easier form to work with, we have that:

$$\begin{aligned} ty' + 2y &= t^2 - t + 1 \\ y' + \frac{2}{t}y &= t - 1 + \frac{1}{t} \end{aligned}$$

We can then say that

$$\begin{aligned} \mu &= e^{\int \frac{2}{t}\, dt} \\ &= e^{2\ln(t)} \\ &= t^2 \end{aligned}$$

So finally

$$
\begin{aligned}
y &= \frac{\int (t^3 - t^2 + t)\, dt + c}{t^2} \\
&= \frac{\frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + c}{t^2} \\
&= \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{c}{t^2}
\end{aligned}
$$

Solving for c we finally get that

$$\boldsymbol{y = \frac{1}{3}t^2 - \frac{1}{3}t + \frac{1}{2}} + \frac{1}{12t^2}$$

Example 3.14. Solve

$$\cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1, \quad y(\pi/4) = 3\sqrt{2}$$

Solution. Let's first rewrite our expression into a easier form to work with, we have that:

$$
\begin{aligned}
\cos(x)y' + \sin(x)y &= 2\cos^3(x)\sin(x) - 1 \\
y' + \tan(x)y &= 2\cos^2(x)\sin(x) - \cos^{-1}(x)
\end{aligned}
$$

We can then say that

$$
\begin{aligned}
\mu &= e^{\int \tan(x)\, dt} \\
&= e^{-\ln|\cos(x)|} \\
&= \cos^{-1}(x)
\end{aligned}
$$

So finally

$$
\begin{aligned}
y &= \frac{\int (2\cos(x)\sin(x) + \cos^{-2}(x))\, dx + c}{\cos^{-1}(x)} \\
&= \frac{\sin^2(x) + \tan(x) + c}{\cos^{-1}(x)} \\
&= \cos(x)\sin^2(x) + \sin(x) + c\cos(x)
\end{aligned}
$$

Solving for c we finally get that

$$\boldsymbol{y = \cos(x)\sin^2(x) + \sin(x) + \frac{9}{2}\cos(x)}$$

3.7 Peano and Picard Theorems

Theorem 3.15. Peano Theorem If there is a rectangle $R = \{(x, y) : x_1 \le x \le x_2 \text{ and } y_1 \le y \le y_2\}$ where $f(x, y)$ is continuous in (x, y), and $(x_0, y_0) \in \mathbb{R}$ (i.e. $y(x_0) = y_0$) then f has a solution in some interval such that $x_0 \in I$ for some $I \subseteq (x_1, x_2)$.

Theorem 3.16. Picard Theorem If f satisfies the criterion for the Peano theorem and also $f(x, y)$ is differentiable in y in R, then the solution in I is also unique.

To summarize Peano theorem tells us that if our differential equation is continuous in some domain then there exists at least one solution on that domain, while the Picard theorem indicates that if our differential equation is both continuous and differential on some domain then there exists only one solution on that domain.

In this class we look at both of these theorems in cohesion:

Theorem 3.17. Existence of a Unique Solution Let R be a rectangular region in the xy-plane defined by $a \leq x \leq b$ and $c \leq y \leq d$ which also contains the point (x_0, y_0). If both $f(x, y)$ and $\partial/\partial y$ are continuous on R, then there exists some interval a solution $y(x)$ on some interval within the bounds of x to the differential equation

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0$$

For example take the differential equation $\frac{dy}{dx} = 3xy^2$. We know this no matter what which rectangle we choose on the reals $f(x, y) = 3xy^2$ is both continuous and differentiable on the reals and as such has a unique solution on whatever domain we specify. One the other hand the differential equations $\frac{dy}{dx} = x|y|$ is continuous on the reals but not differentiable at $y = 0$. So there exist at least one solutions on all the reals but if our domain includes $y = 0$ then we do not have a unique solution.

As a practice for the reader, for each of the examples below determine if the conditions of Peano and Picard theorem are satisfied and conclude what we can determine based on these conditions (as a tip, try to find points at which there might be a discontinutity or "sharp" edges where the function may not be differentiable):

Example 3.18. $R = \{(x, y) : -\pi \leq x \leq \pi, -\pi \leq y \leq \pi\}$, D.E:

$$\frac{dy}{dx} = e^{x^3 \cos(3x) y^2 \sin(y)}$$

$(x_0, y_0) = (0, 0)$

Example 3.19. $R = \{(x, y) : -1 \leq x \leq 4, 2 \leq y \leq 6\}$, D.E:

$$\frac{dy}{dx} = |x - y| + y^{1000}$$

$(x_0, y_0) = (1, 4)$

Example 3.20. $R = \{(x, y) : -2 \leq x \leq 2, 0 \leq y \leq 4\}$, D.E:

$$\frac{dy}{dx} = \begin{cases} xy + e^{x-y}, y \leq 2 \\ xy + e^{x-1}, y < 2 \end{cases}$$

$(x_0, y_0) = (1, 2)$

3.8 Autonomous DEs

To review **Autonomous DEs** are equations of the form $\frac{dy}{dx} = h(y)$ for some function of y, $h(y)$. They are important because we can find interesting properties concerning their potential solutions and the behaviour of their corresponding slope fields.

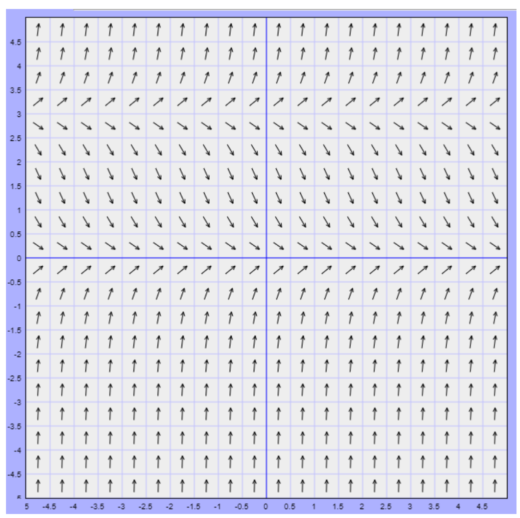
One special property is that values of y that solve the equation $h(y) = 0$ can be considered as constant solutions to the differential equation $\frac{dy}{dx} = h(y)$. The reason why is simple, the derivative of any constant is 0 and also the solution to $h(y)$ at those points are 0 as well. Thus $\frac{dy}{dx} = h(y)$ at such points.

Example 3.21. Find the constant solutions to the differential equation

$$\frac{dy}{dx} = y^2 - 3y$$

Solution. Here we have that $h(y) = y^2 - 3y$. We want to find values of y such that $h(y) = 0$. So we solve $0 = y^2 - 3y$ and can deduce that $y = 0$ and $y = 3$ are constant solutions.

We can also classify these constant solutions based on the behaviour of the slope fields surrounding it. Take the example above, here is its corresponding slope field.



We classify solutions at which the slope field diverges (values of the derivative for y above this solution are negative and below positive) to as **stable** solutions, solutions at which the slope field converges from (values of the derivative for y above this solution are positive and below negative) as **unstable** solutions, and solutions where one side diverges and the other converges (values of the derivative for y above this solution are either both negative or both positive) as **semi-stable** solutions. As such above, $y = 0$ would be a stable solution, whereas $y = 3$ would be an unstable solution.

Recitation 4.

4.9 Population Modeling

There is no inherent model that perfectly describes the behaviour of all types of populations. The basic idea for population dynamics however is that

$$\frac{dP}{dt} = r_i - r_o$$

where r_i is the rate at which the population grows and r_o is the rate at which the population decays/dies.

4.10 Higher Order DE Discussion

A significant portion of what we will discuss through the rest of the semester is solving higher order linear differential equations.

Definition 4.22. We consider a linear ode to be **homogeneous** if it can be written in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_0(x)y = 0$$

We consider a linear ode to be non-homogeneous if it can not be written in the form above.

There are a couple of methods that we will use to solve these types of equations from looking at characteristic polynomials to Laplace transformations.

4.11 Wronskian

Definition 4.23. We classify a set of equations $f_1(x), f_2(x), \ldots, f_n(x)$ as **linearly dependent** if we can form one $f_i(x)$ for $i \in [n]$ as a linear combination of all the other equations in the set. A set of equations that does not have this property is considered **linearly independent**.

The Wronskian is a specific function that can help us determine the independent of a set of functions.

Definition 4.24. Let $y_1(x), y_2(x), \ldots, y_n(x)$ be functions that have at least $n-1$ continuous derivatives. We then define the wronskian as the determinant

$$W(y_1, y_2, \ldots, y_n)(x) := \begin{vmatrix} y_1(x) & y_2(x) & \ldots & y_n(x) \\ y_1'(x) & y_2'(x) & \ldots & y_n'(x) \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \ldots & y_n^{(n-1)}(x) \end{vmatrix}$$

Theorem 4.25. Let $y_1(x), y_2(x), \ldots, y_n(x)$ be functions that have at least $n-1$ continuous derivatives. They are linearly independent on some interval I if $W(y_1, y_2, \ldots, y_n)(x) \neq 0$ for some $x \in I$

4.12 Basic Matrices Review

Theorem 4.26. Let $m, n \in \mathbb{N}$ and define matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \ddots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \dots & \dots & \ddots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{pmatrix}$$

We say that the sum of these two matrices is

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1m} + b_{1m} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2m} + b_{2m} \\ \dots & \dots & \ddots & \dots \\ a_{n1} + b_{n1} & a_{n2} + b_{n2} & \dots & a_{nm} + b_{nm} \end{pmatrix}$$

Theorem 4.27. Let $m, n, k \in \mathbb{N}$ and define matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \ddots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1k} \\ b_{21} & b_{22} & \dots & b_{2k} \\ \dots & \dots & \ddots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mk} \end{pmatrix}$$

We say that the product of these two matrices is

$$A \cdot B = \begin{pmatrix} \sum_i^m a_{1i} b_{i1} & \sum_i^m a_{1i} b_{i2} & \dots & \sum_i^m a_{1i} b_{ik} \\ \sum_i^m a_{2i} b_{i1} & \sum_i^m a_{2i} b_{i2} & \dots & \sum_i^m a_{2i} b_{ik} \\ \dots & \dots & \ddots & \dots \\ \sum_i^m a_{ni} b_{i1} & \sum_i^m a_{ni} b_{i2} & \dots & \sum_i^m a_{ni} b_{ik} \end{pmatrix}$$

Theorem 4.28. For a 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

We have that the inverse of it is

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Theorem 4.29. For a 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

We have that its determinant is

$$|A| = a_{11}a_{22} - a_{12}a_{21}$$

Recitation 5.

5.13 Second Order Homogeneous Linear DEs

Before we get into solving non-homogeneous linear DE's, let's talk about how to solve homogeneous ones (the strategy for solving non-homogeneous is very similar and in fact includes solving a homogeneous DE).

For now, take some differential equation

$$ay'' + by' + cy = 0, \quad a, b, c \in \mathbb{R}$$

We follow a four step process:

1. **Write the characteristic equation.** This is done by substituting y'' for r^2, y' for r, and y for 1.
2. **Solve our new equation** $ar^2 + br + c = 0$ for roots r_1, r_2.
3. **Write our solution.** If $r_1 \neq r_2$ then our final solution is $y = c_1 e^{r_1} + c_2 e^{r_2}$. If $r_1 = r_2$ then our final solution is $y = c_1 e^{r_1} + c_2 x e^{r_2}$ (this is in order to maintain linear independence).

   In addition, take note of Euler's formula: $\cos(x) + i\sin(x) = e^{ix}$ which we use to simplify solutions when the root is a complex number. For example when our roots $r_{1,2} = a \pm ib$ then our corresponding solutions are $y = c_1 e^{ax}\sin(bx) + c_2 e^{ax}\cos(bx)$.
4. **Solve for constants using initial condition.** Since we have two constants that we have to solve for we need at least two initial conditions and form a system of equations. Usually our initial conditions are given in the form $y(k_1) = j_1$ and $y'(k_2) = j_2$. This may seem daunting at first because one initial condition is the derivative of our solution but it is a simple matter of taking the derivative of our general solution and then plugging in our constants.

Let's do a couple practice problems:

Example 5.30. Solve

$$y'' + 11y' + 24y = 0, \quad y(0) = 0, y'(0) = -7$$

Solution. We have the characteristic polynomial $r^2 + 11r + 24 = 0$. This gives us the roots $r_1 = 8$ and $r_2 = 3$ and implies that our general solution is $y(x) = c_1 e^{8x} + c_2 e^{3x}$ which if we take the derivative of gives us $y'(x) = 8c_1 e^{8x} + 3c_2 e^{3x}$. Plugging in our initial conditions we get the system

$$\begin{cases} 0 = c_1 + c_2 \\ -7 = 8c_1 + 3c_2 \end{cases} \implies \begin{cases} c_1 = \frac{7}{5} \\ c_2 = -\frac{7}{5} \end{cases}$$

Thus finally, we have that

$$\boldsymbol{y(x) = \frac{7}{5}e^{8x} - \frac{7}{5}e^{3x}}$$

Example 5.31. Solve

$$y'' - 4y' + 4y = 0, \quad y(0) = 12, y'(0) = -3$$

Solution. We have the characteristic polynomial $r^2 - 4r + 4 = 0$. This gives us the roots $r_{1,2} = 2$ and implies that our general solution is $y(x) = c_1 e^{2x} + c_2 x e^{2x}$ which if we take the derivative of gives us $y'(x) = 2c_1 e^{2x} + 2c_2 x e^{2x} + c_2 e^{2x}$. Plugging in our initial conditions we get the system

$$\begin{cases} 12 = c_1 + c_2 \\ -3 = 2c_1 + c_2 \end{cases} \implies \begin{cases} c_1 = 12 \\ c_2 = -27 \end{cases}$$

Thus finally, we have that

$$\boldsymbol{y(x) = 12e^{2x} - 27xe^{2x}}$$

Example 5.32. Solve

$$y'' + 16y = 0, \quad y\left(\frac{\pi}{2}\right) = -10, y'\left(\frac{\pi}{2}\right) = 3$$

Solution. We have the characteristic polynomial $r^2 + 16 = 0$. This gives us the roots $r_1 = 4i$ and $r_2 = -4i$ and implies that our general solution is $y(x) = c_1 \cos(4x) + c_2 \sin(4x)$ which if we take the derivative of gives us $y'(x) = -4c_1 \sin(x) + 4c_2 \cos(4x)$. Plugging in our initial conditions we get the system

$$\begin{cases} -10 = c_1 \\ 3 = 4c_2 \end{cases} \implies \begin{cases} c_1 = -10 \\ c_2 = \frac{3}{4} \end{cases}$$

Thus finally, we have that

$$\boldsymbol{y(x) = -10\cos(4x) + \frac{3}{4}\sin(4x)}$$

5.14 Mass-Spring Systems

One application of differential equations is systems that involve springs and masses or forces that are exerting on those springs. We typically model three types of systems that involve springs:

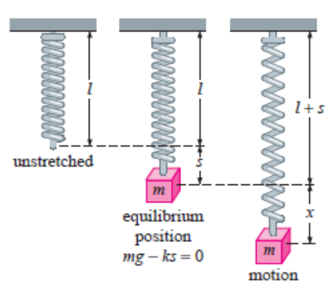
1. **Free Undamped Motion**: we model this with the differential equation

   $$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

   Where $x(t)$ is the position equation with respect to time, k is the spring constant, and m is the mass on the spring. We can even solve for a generalized solution for $x(t)$ where

   $$x(t) = c_1 \cos\left(\frac{k}{m}t\right) + c_2 \sin\left(\frac{k}{m}t\right)$$

   As a visualization we have that this system looks as follows:



2. **Free Damped Motion**: this model is very similar to the previous one except we have that there is an extra external force that dampens the movement of the spring (for example, have a mass spring system underwater or some other more viscous liquid). We can model this system with the equation

$$\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

where all the variables are as defined above and β is a positive damping constant. There are three cases of potential general solutions based on the values of λ^2 and ω^2.

   (a) Overdamped motion: $\boldsymbol{\lambda^2 - \omega^2 > 0 \implies x(t) = e^{-\lambda t}\left(c_1 e^{\sqrt{\lambda^2-\omega t}} + c_2 e^{-\sqrt{\lambda^2-\omega^2}t}\right)}$

   (b) Critically damped motion: $\boldsymbol{\lambda^2 - \omega^2 = 0 \implies x(t) = e^{-\lambda t}(c_1 + c_2 t)}$

   (c) Underdamped motion: $\boldsymbol{\lambda^2 - \omega^2 < 0 \implies x(t) = e^{-\lambda t}\left(c_1 \cos\left(\sqrt{\omega^2-\lambda^2}t\right) + c_2 \sin\left(\sqrt{\omega^2-\lambda^2}t\right)\right)}$

3. **Driven Motion**: this model is also similar to the last model except instead of a damping force we have another force that influences our model by "moving the spring faster". For the same variable definitions above and some function $f(t)$ which is the external force that acting on the vibrating mass on the string we have that our differential equation modeling this behavious can be represented by

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = \frac{f(t)}{m}$$

Recitation 6.

6.15 Undetermined Coefficients

So far we have only talked about solving higher order differential equations when they are homogeneous. More often that not, however, there will be some expression on the right hand side that is non zero. Remember, in the context of higher order linear DE's a non-homogeneous equation is one of the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_0(x)y = g(x)$$

for some non zero function $g(x)$.

The strategy we use to solve these problems is to break it down into parts:

1. Solve the "left hand side" by setting $g(x) = 0$ and using the characteristic polynomial as we do with homogeneous equations. We call this the complementary solution.

2. Use the method on undetermined coefficients on the "right hand side". This method is based on the classification of $g(x)$. We do this by determining a "guess solution" and then plugging that into the left hand side expression and then solving for any coefficients. Here is a comprehensive list of potential $g(x)$ equations and corresponding guess solutions you may encounter:

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x+7$	$Ax+B$
3. $3x^2-2$	Ax^2+Bx+C
4. x^3-x+1	Ax^3+Bx^2+Cx+E
5. $\sin 4x$	$A\cos 4x+B\sin 4x$
6. $\cos 4x$	$A\cos 4x+B\sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x-2)e^{5x}$	$(Ax+B)e^{5x}$
9. x^2e^{5x}	$(Ax^2+Bx+C)e^{5x}$
10. $e^{3x}\sin 4x$	$Ae^{3x}\cos 4x+Be^{3x}\sin 4x$
11. $5x^2\sin 4x$	$(Ax^2+Bx+C)\cos 4x+(Ex^2+Fx+G)\sin 4x$
12. $xe^{3x}\cos 4x$	$(Ax+B)e^{3x}\cos 4x+(Cx+E)e^{3x}\sin 4x$

We call this the particular solution.

3. Sum up the complementary and particular solution to get our final solution. Solve for any remaining constants using the initial conditions.

Talking about it at a higher level may seem a bit abstract so let's look at a couple of examples.

Example 6.33. Solve the following IVP

$$y''-4y'-12y=3e^{5t}, \quad y(0)=\frac{18}{7}, \quad y'(0)-\frac{1}{7}$$

Solution. Let's follow the steps we have outlined above:

1. Let's start by making the characteristic polynomial $r^2-4r-12=0$. We then have that our roots are $r_1=6, r_2=-2$ and the corresponding complementary solution is $y_c=c_1e^{6t}+c_2e^{-2t}$
2. Since we have that $g(x)=3e^{5t}$ we know that our guess solution for the particular solution is of the form $y_p=Ae^{5t}$. Plugging this back in to the expression in order to solve for A we have that $25Ae^{5t}-20A^{5t}-12Ae^{5t}=3e^{5t} \implies A=-\frac{3}{7}$. So we have that $y_p=-\frac{3}{7}e^{5t}$.
3. And so with these two parts our final answer is now $y=y_c+y_p=c_1e^{6t}+c_2e^{-2t}-\frac{3}{7}e^{5t}$. Since we have initial conditions we would plug them in now and get the final answer

$$\boldsymbol{y=2e^{-2t}+e^{6t}-\frac{3}{7}e^{5t}}$$

Example 6.34. Solve the following DE

$$4y''+y=6t\cos\left(\frac{t}{2}\right)$$

Solution. Let's follow the steps we have outlined above

1. First, start with the characteristic polynomial: $4r^2+1=0$ which has corresponding roots $r_{1,2}=\pm\frac{1}{2}i$. This gives us the complementary solution $y_c=c_1\sin\left(\frac{t}{2}\right)+c_2\cos\left(\frac{t}{2}\right)$

2. Since we have that $g(t) = 6t\cos\left(\frac{t}{2}\right)$ we know that our guess solution for the particular solution is of the form $y_p = A\cos\left(\frac{t}{2}\right) + B\sin\left(\frac{t}{2}\right)$. By plugging this back into our left hand side we then get that $B = \frac{3}{2}$.

3. We then get the answer $y = y_c + y_p = c_1\sin\left(\frac{t}{2}\right) + c_2\cos\left(\frac{t}{2}\right) + \frac{3}{2}\sin\left(\frac{t}{2}\right)$. Since, however, in our complementary solution we have a term that is linearly dependent with a term in our particular solution we should multiply the $\frac{3}{2}\sin\left(\frac{t}{2}\right)$ by a t which gives us the final answer

$$\boldsymbol{y = c_1 \sin\left(\frac{t}{2}\right) + c_2\cos\left(\frac{t}{2}\right) + \frac{3}{2}t\sin\left(\frac{t}{2}\right)}$$

We will leave the last one as an exercise.

Example 6.35. Solve the following DE

$$4y'' - 16y' - 17y = e^{-2t}\sin\left(\frac{t}{2}\right) + 6t\cos\left(\frac{t}{2}\right)$$

6.16 Introduction to Laplace

Laplace transforms are a really powerful method of solving differential equations that include certain functions. That being said there is a lot of content to cover regarding them and so we shall first start with some definitions.

Definition 6.36. Let $f(t)$ be a function defined for $t \geq 0$. Then the **Laplace transform** of f (given that the integral converges) is

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

We can tell based on this definition alone that the Laplace transform has certain properties such as linearity. More specifically we have that for some functions $f_1(x), f_2(x)$ and constants $c_1, c_2 \in \mathbb{R}$ we have that $\mathcal{L}\{c_1 f_1(x) + c_2 f_2(x)\} = c_1\mathcal{L}\{f_1(x)\} + c_2\mathcal{L}\{f_2(x)\}$.

Above remember that we said this transform only works on certain functions. This is because we have a infinite integral and we want to ensure that this value converges. For simplicity's sake, however, we have that

Definition 6.37. A function f is said to be of **exponential order** if there exist constants $c, M > 0$, and $T > 0$ such that $|f(t)| \leq Me^{ct}$ for all $t > T$.

and also

Theorem 6.38. If f is piecewise continuous on $[0, \infty)$ and of exponential order, then $\mathcal{L}\{f(t)\}$ exists for $s > c$.

With all this being said let's take the Laplace transform of a couple of basic functions

Example 6.39. Evaluate $\mathcal{L}\{1\}$

Solution. Using the definition we have that

$$\begin{aligned}
\mathcal{L}\{1\} &= \int_0^\infty e^{-st}(1)\,dt \\
&= \lim_{b\to\infty} \int_0^b e^{-st}\,dt \\
&= \lim_{b\to\infty} \left.\frac{-e^{-st}}{s}\right|_0^b \\
&= \lim_{b\to\infty} \frac{-e^{-sb}+1}{s} \\
&= \boldsymbol{\frac{1}{s}}
\end{aligned}$$

Example 6.40. Evaluate $\mathcal{L}\{t\}$

Solution. Using the definition we have that

$$\begin{aligned}
\mathcal{L}\{1\} &= \int_0^\infty e^{-st}(t)\,dt \\
&= \lim_{b\to\infty} \left(\left.\frac{-te^{-st}}{s}\right|_0^b + \frac{1}{s}\int_0^b e^{-st} \right) dt \\
&= \frac{1}{s}\mathcal{L}\{1\} \\
&= \boldsymbol{\frac{1}{s^2}}
\end{aligned}$$

Here is a table of a lot of the transformations you will encounter in this class:

$f(t)=\mathcal{L}^{-1}\{F(s)\}$	$F(s)=\mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$t^n, \quad n=1,2,3,...$	$\frac{n!}{s^{n+1}}$
$t^p, \quad p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$
$t^{n-\frac{1}{2}}, \quad n=1,2,3,...$	$\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$
$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$
$\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
$\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$
$\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$
$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$
$e^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
$t^n e^{at}, \quad n=1,2,3,...$	$\frac{n!}{(s-a)^{n+1}}$
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
$e^{ct}f(t)$	$F(s-c)$
$t^n f(t), \quad n=1,2,3,...$	$(-1)^n F^{(n)}(s)$
$\frac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f'(0)-\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$

Recitation 7.

7.17 Laplace: Solving IVPs

For solving a differential equation using laplace transforms we basically go through a four step process:

1. **Perform a laplace transform of both sides of our differential equation**. For example, if we were to have the equation $ay''+by'+cy = g(x)$ we would then perform the transformation $\mathcal{L}(ay''+by'+cy) = \mathcal{L}(g(x))$.
2. **Plug in any initial conditions and solve for $Y(s)$**. This is just simple algebraic manipulation.
3. **Use partial fractions to breakup our expression for $Y(s)$ into components that we can do an inverse laplace transform on**. This again is algebraic manipulation but also is a matter of practice so that you can recognize what an expressions are easy to take the inverse laplace transform on.
4. **Take the inverse laplace transform**.

To better contextualize this process, let's do a couple of examples.

Example 7.41. Solve the following IVP:

$$y'' - 10y' + 9y = 5t, \quad y(0) = -1, \quad y'(0) = 2$$

Solution. Let's approach this using the four steps we have outlined above:

1. First, take the transform of every term in the DE:

   $$\mathcal{L}\{y''\} - 10\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{5t\}$$

   Use the table to get:

   $$s^2Y(s) - sy(0) - y'(0) - 10(sY(s) - y(0)) + 9Y(s) = \frac{5}{s^2}$$

2. Plug in the initial conditions and combine terms that have a $Y(s)$ in them to get:

   $$(s^2 - 10s + 9)Y(s) + s - 12 = \frac{5}{s^2}$$

   Solve for $Y(s)$:

   $$Y(s) = \frac{5}{s^2(s-9)(s-1)} + \frac{12-s}{(s-9)(s-1)}$$

3. In order to find the solution, we need to take the inverse transform. Let's combine the two terms:

   $$Y(s) = \frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)}$$

The partial fraction decomposition for this transform is

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}$$

Setting numerators equal gives

$$5 + 12s^2 + s^3 = As(s-9)(s-1) + B(s-9)(s-1) + Cs^2(s-1) + Ds^2(s-9)$$

Skipping all the solving parts, we get

$$A = \frac{50}{81},\ B = \frac{5}{9},\ C = \frac{31}{81},\ D = -2$$

Plugging in the constants gives

$$Y(s) = \frac{\frac{50}{81}}{s} + \frac{\frac{5}{9}}{s^2} + \frac{\frac{31}{81}}{s-9} - \frac{2}{s-1}$$

4. Finally, taking the inverse transform gives us the solution:

$$\boldsymbol{y(t) = \frac{50}{81} + \frac{5}{9}t + \frac{31}{81}e^{9t} - 2e^t}$$

Example 7.42. Solve the following IVP:

$$2y'' + 3y' - 2y = te^{-2t}, \quad y(0) = 0, \quad y'(0) = -2$$

Solution. Let's approach this using the four steps we have outlined above:

1. Take the Laplace transform of all the terms and plug in initial conditions to get:

$$2(s^2Y(s) - sy(0) - y'(0)) + 3(sY(s) - y(0)) - 2Y(s) = \frac{1}{(s+2)^2}$$
$$(2s^2 + 3s - 2)Y(s) + 4 = \frac{1}{(s+2)^2}$$

2. Solve for $Y(s)$:

$$Y(s) = \frac{1}{(2s-1)(s+2)^3} - \frac{4}{(2s-1)(s+2)}$$

3. Combine together so partial fractions won't be as bad:

$$\begin{aligned} Y(s) &= \frac{1 - 4(s+2)^2}{(2s-1)(s+2)^3} \\ &= \frac{-4s^2 - 16s - 15}{(2s-1)(s+2)^3} \end{aligned}$$

The partial fraction decomposition is:

$$Y(s) = \frac{A}{2s-1} + \frac{B}{s+2} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)^3}$$

Solving, we get:

$$A = -\frac{192}{125},\ B = \frac{96}{125},\ C = -\frac{2}{25},\ D = -\frac{1}{5}$$

Plugging these into the transform and factoring out a $\frac{1}{125}$ gives:

$$Y(s) = \frac{1}{125}\left(-\frac{192}{2\left(s-\frac{1}{2}\right)} + \frac{96}{s+2} - \frac{10}{(s+2)^2} - \frac{25\frac{2!}{2!}}{(s+2)^3}\right)$$

4. Taking the inverse transform gives:

$$\boldsymbol{y(t) = \frac{1}{125}\left(-96e^{\frac{t}{2}} + 96e^{-2t} - 10te^{-2t} - \frac{25}{2}t^2e^{2t}\right)}$$

Example 7.43. Solve the following IVP:

$$y'' - 6y' + 15y = 2\sin(3t), \quad y(0) = -1, \quad y'(0) = -4$$

Solution. Again, the four steps:

1. Take the Laplace transform of everything and plug in the initial conditions to get:

$$(s^2 - 6s + 15)Y(s) + s - 2 = \frac{6}{s^2+9}$$

2. Solve for $Y(s)$ and combine into a single term:

$$Y(s) = \frac{-s^3 + 2s^2 - 9s + 24}{(s^2+9)(s^2-6s+15)}$$

3. The partial fraction decomposition is:

$$Y(s) = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2-6s+15}$$

Solving gives us:

$$A = \frac{1}{10},\ B = \frac{1}{10},\ C = -\frac{11}{10},\ D = \frac{5}{2}$$

Plug these back in, complete the square on the denominator of the second term, and adjust the numerators:

$$\begin{aligned}
Y(s) &= \frac{1}{10}\left(\frac{s+1}{s^2+9} + \frac{-11s+25}{s^2-6s+15}\right) \\
&= \frac{1}{10}\left(\frac{s+1}{s^2+9} + \frac{-11(s-3+3)+25}{(s-3)^2+6}\right) \\
&= \frac{1}{10}\left(\frac{s}{s^2+9} + \frac{1\frac{3}{3}}{s^2+9} - \frac{11(s-3)}{(s-3)^2+6} - \frac{8\frac{\sqrt{6}}{\sqrt{6}}}{(s-3)^2+6}\right)
\end{aligned}$$

4. Take the inverse transform:

$$\boldsymbol{y(t) = \frac{1}{10}\left(\cos(3t) + \frac{1}{3}\sin(3t) - 11e^{3t}\cos(\sqrt{6}t) - \frac{8}{\sqrt{6}}e^{3t}\sin(\sqrt{6}t)\right)}$$

Recitation 8.

8.18 Laplace: Translation on s-Axis

Despite all the Laplace transformations we've already discussed there are still a couple of ways that we can do transformations. Specifically the next couple of sections will be dedicated to looking at translating the functions we already know.

Theorem 8.44. Translation on the s-Axis: If $\mathcal{L}\{f(t)\} = F(s)$ and a is any real number, then

$$\boldsymbol{\mathcal{L}\{e^{at}f(t)\} = F(s-a)}$$

This concept isn't too difficult to grasp so we will just do one example

Example 8.45. Solve

$$y'' - 6y' + 9y = t^2e^{3t}, \quad y(0) = 2, y'(0) = 17$$

Solution. As always, we follow our four steps:

1. **Take the Laplace Transform**: We have that

$$\begin{aligned}\mathcal{L}\{y'' - 6y' + 9y\} &= \mathcal{L}\{t^2e^{3t}\}\\ s^2Y(s) - sy(0) - y'(0) - 6[sY(s) - y(0)] + 9Y(s) &= \frac{2}{(s-3)^2}\end{aligned}$$

2. **Plug in initial conditions and solve for Y**: We have that

$$\begin{aligned}s^2Y(s) - sy(0) - y'(0) - 6[sY(s) - y(0)] + 9Y(s) &= \frac{2}{(s-3)^3}\\ s^2Y(s) - 2s - 17 - 6sY(s) - 12 + 9Y(s) &= \frac{2}{(s-3)^3}\\ (s^2 - 6s + 9)Y(s) &= 2s + 5 + \frac{2}{(s-3)^2}\\ Y(s) &= \frac{2s+5}{(s-3)^2} + \frac{2}{(s-3)^5}\end{aligned}$$

3. **Rewrite terms in order to take inverse Laplace transform**: The second term in the expression above is already fine so let's do partial fractions on the first term

$$\begin{aligned}\frac{2s+5}{(s-3)^2} &= \frac{A}{s-3} + \frac{B}{(s-3)^2}\\ As - 3A + B &= 2s + 5\\ A &= 2\\ B &= 11\end{aligned}$$

So finally we have that

$$Y(s) = \frac{2}{s-3} + \frac{11}{(s-3)^2} + \frac{2}{(s-3)^5}$$

4. **Take the inverse Laplace transform**: We have that

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{\frac{2}{s-3} + \frac{11}{(s-3)^2} + \frac{2}{(s-3)^5}\}$$
$$\boldsymbol{y(t) = 2e^{3t} + 11te^{3t} + \frac{1}{12}t^4e^{3t}}$$

8.19 Laplace: Translation on t-Axis

Similar to a translation on the s-Axis, we can also do translations on the t-Axis. We achieve this by using Unit step functions (also known as heavyside functions).

Definition 8.46. The **unit step function** $\mathcal{U}(t-a)$ is defined to be

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

It comes naturally then that

$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases} = g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a)$$

and also

$$f(t) = \begin{cases} 0, & 0 \le t < a \\ g(t), & a \le t < b \\ 0, & t \ge b \end{cases} = g(t)[\mathcal{U}(t-a) - \mathcal{U}(t-b)].$$

With these two facts we can form the majority of piece-wise functions that seem to take on different function trends on different intervals.

Example 8.47. Write the piecewise function $f(t) = \begin{cases} 20t, & 0 \le t < 5 \\ 0, & t \ge 5 \end{cases}$ in terms of Unit Step functions.

Solution. As expressed above, we already know the general form of a piecewise function with two different cases is $f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases} = g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a)$. In this case right now we have that $a = 5$, $g(t) = 20t$, and $h(t) = 5$ which then implies that our form using unit step functions is finally

$$\boldsymbol{f(t) = 20t - 20t\mathcal{U}(t-5)}$$

So far, we haven't really talked about Laplace transforms, now let's bring it in all together forgoing any proofs for now.

Theorem 8.48. If $F(s) = \mathcal{L}\{f(t)\}$ and $a > 0$, then

$$\boldsymbol{\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)}$$

Sometimes we will encounter equations in others forms and so we also have the equivalent expression

$$\boldsymbol{\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}}$$

With all this in mind, let's now do a practice problem

Example 8.49. Solve $y' + y = f(t), \quad y(0) = 5$, where $f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\cos(t), & t \ge \pi \end{cases}$

Solution. Let's first translate our piece-wise function into an expression in terms of unit step functions. Based on the work we've done above we can say clearly that $f(t) = 3\cos(t)\mathcal{U}(t-\pi)$. Our new problem to solve is now $y' + y = 3\cos(t)\mathcal{U}(t-\pi)$.

We have that

$$\begin{aligned} \mathcal{L}\{y' + y\} &= \mathcal{L}\{3\cos(t)\mathcal{U}(t-\pi)\} \\ sY(s) - y(0) + Y(s) &= \frac{-3s}{s^2+1}e^{-\pi s} \\ (s+1)Y(s) - 5 &= \frac{-3s}{s^2+1}e^{-\pi s} \\ (s+1)Y(s) &= 5 + \frac{-3s}{s^2+1}e^{-\pi s} \\ Y(s) &= \frac{5}{s+1} + \frac{-3s}{s^2+1}e^{-\pi s} \end{aligned}$$

Taking the inverse Laplace transform we finally have that

$$\boldsymbol{y(t) = 5e^{-t} + \frac{3}{2}[e^{-(t-\pi)} + \sin(t) + \cos(t)]\mathcal{U}(t-\pi)}$$

8.20 Laplace: Impulse and the Dirac Delta Function

The Dirac Delta function is typically a very confusing subject topic to gain an intuition about. For the purpose of this class the mechanics of the proof and uses of the Dirac Delta function are not as relevant as the special properties of it in the context of Laplace transforms.

Definition 8.50. We define the **unit impulse** function as the function

$$\boldsymbol{\delta_a(t-t_0) = \begin{cases} 0, & 0 \le t < t_0 - a \\ \frac{1}{2a}, & t_0 - a \le t < t_0 + a \\ 0, & t \ge t_0 + a \end{cases}}$$

Note that this has the special property that $\int_0^\infty \delta_a(t-t_0)\,dt = 1$ which makes intuitive sense because if we think of the integral as the area under the curve we then have that the value of the integral is $2a \cdot \frac{1}{2a} = 1$.

Definition 8.51. We define the **Dirac Delta Function** as the "function" δ (it's not really a function because it involves a limit) such that

$$\boldsymbol{\delta(t-t_0) = \lim_{a \to 0} \delta_a(t-t_0)}$$

Finally, forgoing a proof again (if you would like one, email me or just think about using both the definition of a Laplace transform and the definition of the Dirac Delta)

Theorem 8.52. For $t_0 > 0$.

$$\boldsymbol{\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}}$$

Example 8.53. Solve

$$y' - 3y = \delta(t-2), \quad y(0) = 0$$

Solution. We have that

$$\begin{aligned}
\mathcal{L}\{y' - 3y\} &= \mathcal{L}\{\delta(t-2)\} \\
sY(s) - y(0) - 3Y(s) &= e{-}2s \\
(s-3)Y(s) &= e^{-2s} \\
Y(s) &= \frac{e^{-2s}}{s-3} \\
\boldsymbol{y(t)} &\boldsymbol{= e^{3(t-2)}\mathcal{U}(t-2)}
\end{aligned}$$

Note here we have both a translation on the t-axis and use the Dirac Delta function.

Example 8.54. Solve

$$y'' + y = \delta(t-2\pi), \quad y(0) = 0, \quad y'(0) = 1$$

Solution. We have that

$$\begin{aligned}
\mathcal{L}\{y'' + y\} &= \mathcal{L}\{\delta(t-2\pi)\} \\
sY(s) - sy(0) - y'(0) + Y(s) &= e{-}2\pi s \\
(s^2+1)Y(s) &= e^{-2\pi s} + 1 \\
Y(s) &= \frac{e^{-2\pi s}}{s^2+1} + \frac{1}{e^2+1} \\
y(t) &= \sin(t) + \sin(t-2\pi)\mathcal{U}(t-2\pi) \\
\boldsymbol{y(t)} &\boldsymbol{= \sin(t) + \sin(t)\mathcal{U}(t-2\pi)}
\end{aligned}$$

Recitation 9.

9.21 Systems: Generalized Approach

Often times in the real world, differential equations are intrinsically tied. Similar to regular systems of equations that we are used to from high school algebra we have a couple methods to deal with solving these types of questions.

In this class we will use matrices to help solve this system. If you remember back to your linear algebra days we do this by looking at the eigenvalues of the coefficient matrix. As such, the generalized approach to solving systems of differential equations is as follows, we will work with a system of two differential equations

$$\begin{cases} ax'(t) + by'(t) = j \\ cx'(t) + dy'(t) = k \end{cases}$$

Where we are trying to solve for functions $x(t)$ and $y(t)$ which we can also rewrite as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} j(t) \\ k(t) \end{bmatrix}$$

The process of solving these is as follows and is broken into two parts, first is solving the complementary solutions.

1. Solve for the eigenvalues of the coefficient matrix λ_1, λ_2
2. Plug eigenvalues back into the characteristic polynomial and solve for the eigenvectors η_1, η_2
3. Your complementary solutions are then of the form

$$\begin{bmatrix} x_c \\ y_c \end{bmatrix} = \vec{\eta}_1 e^{\lambda_1 t} + \vec{\eta}_2 e^{\lambda_2 t}$$

The process is the same for systems of equations with 3 or more equations. To solve for the particular solutions, simply use the method of undetermined coefficients on each row of the matrices.

9.22 Systems: Real Eigenvalues

Consider the system

$$\vec{x}' = A\vec{x}$$

which has solutions of the form

$$\vec{x} = \vec{\eta} e^{\lambda t}$$

where λ and $\vec{\eta}$ are eigenvalues and eigenvectors of the matrix A. We will be looking at 2×2 systems with two solutions $\vec{x}_1(t)$ and $\vec{x}_2(t)$ where the determinant of the matrix

$$X = (\vec{x}_1 \ \vec{x}_2)$$

is nonzero.

We will start with the case where the two eigenvalues λ_1 and λ_2 are real and distinct. The general solution is

$$\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{\eta}^{(1)} + c_2 e^{\lambda_2 t} \vec{\eta}^{(2)}$$

Tips for sketching phase planes:

- Negative eigenvalues $\implies$ trajectories will move in towards the origin as t increases.
- Positive eigenvalues $\implies$ trajectories will move away from the origin as t increases

- For two negative eigenvalues, assume $\lambda_2 > \lambda_1$. For large and positive t's this means that the solution for λ_2 will be smaller than the solution for λ_1. Therefore, as t increases, the trajectory will move in towards the origin parallel to $\vec{\eta}^{(1)}$. Since the $\lambda_2 > \lambda_1$, the solution for λ_2 will dominate for large and negative t's. Therefore, as we decrease t, the trajectory will move away from the origin parallel to $\vec{\eta}^{(2)}$.

Example 9.55. Solve the following IVP and sketch the phase portrait for the system.

$$\vec{x}' = \begin{pmatrix} -5 & 1 \\ 4 & -2 \end{pmatrix} \vec{x}, \quad \vec{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Solution. Find eigenvalues:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -5-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} \\ &= \lambda^2 + 7\lambda + 6 \\ &= (\lambda+1)(\lambda+6) \implies \lambda_1 = -1,\ \lambda_2 = -6 \end{aligned}$$

For $\lambda_1 = -1$:

$$\begin{pmatrix} -4 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies -4\eta_1 + \eta_2 = 0 \implies \eta_2 = 4\eta_1$$

The eigenvector in this case is

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ 4\eta_1 \end{pmatrix} \implies \vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad \eta_1 = 1$$

For $\lambda_2 = -6$:

$$\begin{pmatrix} 1 & 1 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \eta_1 + \eta_2 = 0 \implies \eta_1 = -\eta_2$$

The eigenvector in this case is

$$\vec{\eta} = \begin{pmatrix} -\eta_2 \\ \eta_2 \end{pmatrix} \implies \vec{\eta}^{(2)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \eta_2 = 1$$

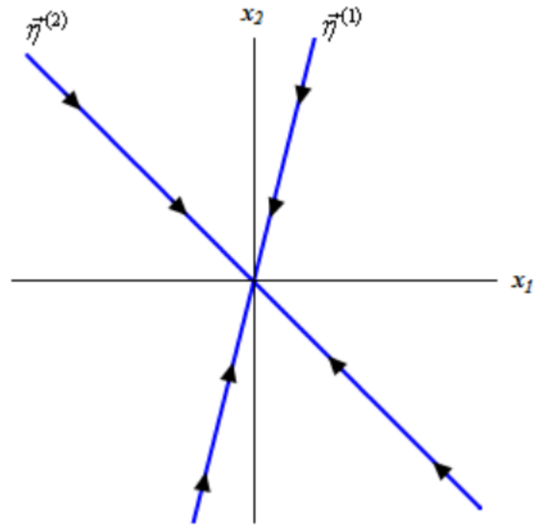
The general solution is:

$$\vec{x}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 4 \end{pmatrix} + c_2 e^{-6t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

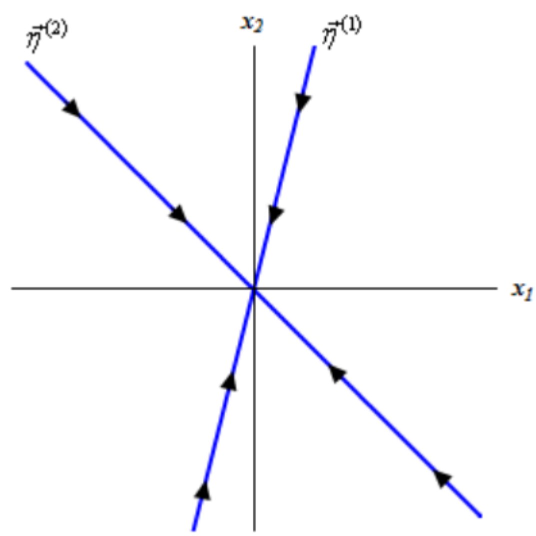
Solving for the constants by plugging in initial conditions, we get that the solution is:

$$\boldsymbol{\vec{x}(t) = \frac{3}{5} e^{-t} \begin{pmatrix} 1 \\ 4 \end{pmatrix} - \frac{2}{5} e^{-6t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

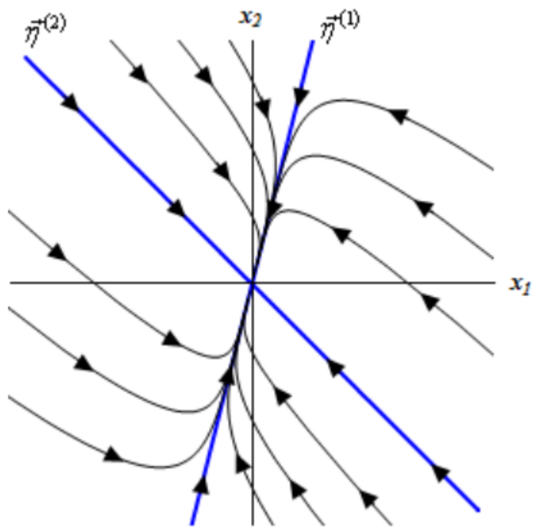
First sketch the trajectories corresponding to the eigenvectors. Both eigenvalues are negative so trajectories for these will move in towards the origin as t increases. When we sketch the trajectories, we will add in arrows to denote the direction they take as t increases.



The second eigenvalue is larger than the first. For large and positive t's this means that the solution for the second eigenvalue will be smaller than the solution for the first. Therefore, as t increases, the trajectory will move in towards the origin parallel to $\vec{\eta}^{(1)}$. Since the second eigenvalue is larger than the first, this solution will dominate for large and negative t's. Therefore, as we decrease t, the trajectory will move away from the origin parallel to $\vec{\eta}^{(2)}$.



In these cases, we call the equilibrium solution $(0, 0)$ a node and it is asymptotically stable. Equilibrium solutions are asymptotically stable if all the trajectories move in toward it as t increases.



Recitation 10.

10.23 Systems: Repeated Eigenvalues

We want two linearly independent solutions so that we can form a general solution. However, with a double eigenvalue we will have only one,

$$\vec{x}_1 = \vec{\eta} e^{\lambda t}$$

So, we need to come up with a second solution. Skipping the derivation stuff,

$$\vec{x}_2 = t e^{\lambda t} \vec{\eta} + e^{\lambda t} \vec{\rho}$$

will be a solution to the system provided $\vec{\rho}$ is a solution to

$$(A - \lambda I)\vec{\rho} = \vec{\eta}$$

Note that this solution and the first solution are linearly independent so they form a fundamental set of solutions and so the general solution in the double eigenvalue case is:

$$\vec{x}(t) = c_1 e^{\lambda t} \vec{\eta} + c_2 (t e^{\lambda t} \vec{\eta} + e^{\lambda t} \vec{\rho})$$

In phase planes, sometimes you will hear nodes for the repeated eigenvalue case called **degenerate nodes** or **improper nodes.**

Example 10.56. Solve the following IVP and sketch the phase portrait for the system.

$$\vec{x}' = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} \vec{x}, \quad \vec{x}(0) = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

Solution. First, find eigenvalues:

$$\begin{aligned}\det(A-\lambda I) &= \begin{vmatrix} 7-\lambda & 1 \\ -4 & 3-\lambda \end{vmatrix} \\ &= \lambda^2 - 10\lambda + 25 \\ &= (\lambda-5)^2 \implies \lambda = 5\end{aligned}$$

This is a double eigenvalue. Let's find the eigenvector:

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies 2\eta_1 + \eta_2 = 0 \implies \eta_2 = -2\eta_1$$

The eigenvector is then:

$$\begin{aligned}\vec{\eta} &= \begin{pmatrix} \eta_1 \\ -2\eta_1 \end{pmatrix} \quad \eta_1 \neq 0 \\ \vec{\eta}^{(1)} &= \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \eta_1 = 1\end{aligned}$$

The next step is to find $\vec{\rho}$. We need to solve:

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \implies 2\rho_1 + \rho_2 = 1 \implies \rho_2 = 1 - 2\rho_1$$

Note that this is almost identical to the system that we solve to find the eigenvalue. The only difference is the RHS. The most general possible $\vec{\rho}$ is:

$$\begin{pmatrix} \rho_1 \\ 1-2\rho_1 \end{pmatrix} \implies \vec{\rho} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{if } \rho_1 = 0$$

In this case, unlike the eigenvector system, we can choose the constant to be anything we want, so we might as well pick it to make our life easier. This usually means picking it to be 0.

The general solution is:

$$\vec{x}(t) = c_1 e^{5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2\left(e^{5t}t\begin{pmatrix} 1 \\ -2 \end{pmatrix} + e^{5t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

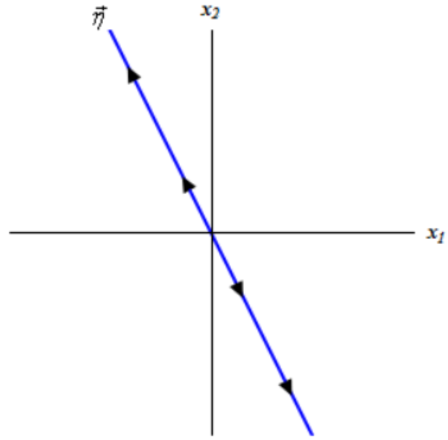
Applying the initial condition to find the constants gives us:

$$\begin{pmatrix} 2 \\ -5 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} \implies c_1 = 2,\ c_2 = -1$$

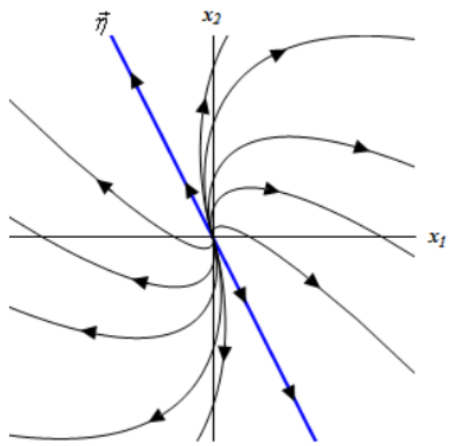
The actual solution is:

$$\begin{aligned}\vec{x}(t) &= 2e^{5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} - \left(te^{5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + e^{5t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\ \boldsymbol{\vec{x}(t)} &\boldsymbol{= e^{5t}\begin{pmatrix} 2 \\ -5 \end{pmatrix} - e^{5t}t\begin{pmatrix} 1 \\ -2 \end{pmatrix}}\end{aligned}$$

We'll first sketch in a trajectory that is parallel to the eigenvector and note that since the eigenvalue is positive the trajectory will be moving away from the origin.



Here's the full thing:



Trajectories in these cases always emerge from (or move into) the origin in a direction that is parallel to the eigenvector. Likewise, they will start in one direction before turning around and moving off into the other direction. The directions in which they move are opposite depending on which side of the trajectory corresponding to the eigenvector we are on. Also, as the trajectories move away from the origin it should start becoming parallel to the trajectory corresponding to the eigenvector.

So, how do we determine the direction? We can do the same thing that we did in the complex case. We'll plug in $(1, 0)$ into the system and see which direction the trajectories are moving at that point. Since this point is directly to the right of the origin the trajectory at that point must have already turned around and so this will give the direction that it will traveling after turning around.

Doing that for this problem to check our phase portrait gives:

$$\begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$$

This vector will point down into the fourth quadrant and so the trajectory must be moving into the fourth quadrant as well. This does match up with our phase portrait.

In these cases, the equilibrium is called a node and is unstable in this case.

10.24 Systems: Complex Eigenvalues

Now we will look at solutions to

$$\vec{x}' = A\vec{x}$$

where the eigenvalues of A are complex. We are going to approach this in a similar way as we did for second order DEs. We want our solutions to only have real numbers in them, however since our solutions to systems are of the form

$$\vec{x} = \vec{\eta}e^{\lambda t}$$

we are going to have complex numbers come into our solution from both the eigenvalue and the eigenvector. Getting rid of the complex numbers here will be similar to how we did it back in second order DEs but will require a little more work.

Recall Euler's formula (we will need it):

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}$$

The general solution to a system with complex roots is

$$\vec{x}(t) = c_1\vec{u}(t) + c_2\vec{v}(t)$$

where $\vec{u}(t)$ and $\vec{v}(t)$ are found by writing the solution as

$$\vec{x}(t) = \vec{u}(t) + i\vec{v}(t)$$

Phase portraits:

- When the eigenvalues are purely imaginary, the trajectories of the solutions will be circles or ellipses that are centered at the origin.
- The only thing that we really need to concern ourselves with are whether they are rotating in a clockwise or counterclockwise direction. We can do this by picking a value of $\vec{x}(t)$ and plugging it into the system to get a vector that will be tangent to the trajectory at that point and pointing in the direction that the trajectory is traveling.

This all probably sounds very vague, so go ahead to the examples section.

Example 10.57. Solve the following IVP and sketch the phase portrait for the general system:

$$\vec{x}' = \begin{pmatrix} 3 & 9 \\ -4 & -3 \end{pmatrix} \vec{x}, \quad \vec{x}(0) = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

Solution. First, find the eigenvectors:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 3-\lambda & 9 \\ -4 & -3-\lambda \end{vmatrix} \\ &= \lambda^2 + 27 \implies \lambda_{1,2} = \pm 3\sqrt{3}i \end{aligned}$$

We only need to get the eigenvector for one of the eigenvalues since we can get the second eigenvector for free from the first one.

For $\lambda_1 = 3\sqrt{3}i$, we need to solve the following system:

$$\begin{pmatrix} 3-3\sqrt{3}i & 9 \\ -4 & -3-3\sqrt{3}i \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Using the first equation, we get

$$\begin{aligned} (3-3\sqrt{3}i)\eta_1 + 9\eta_2 &= 0 \\ \eta_2 &= -\frac{1}{3}(1-\sqrt{3}i)\eta_1 \end{aligned}$$

So the first eigenvector is

$$\begin{aligned} \vec{\eta} &= \begin{pmatrix} \eta_1 \\ -\frac{1}{3}(1-\sqrt{3}i)\eta_1 \end{pmatrix} \\ \vec{\eta}^{(1)} &= \begin{pmatrix} 3 \\ -1+\sqrt{3}i \end{pmatrix} \quad \eta_1 = 3 \end{aligned}$$

When finding the eigenvectors in these cases, make sure the complex number appears in the numerator of any fractions since we'll need it in the numerator later on. Also try to clear out any fractions by appropriately picking the constant to make our lives easier.

The second eigenvector is:

$$\vec{\eta}^{(2)} = \begin{pmatrix} 3 \\ -1-\sqrt{3}i \end{pmatrix}$$

Spoiler alert: We actually won't need this eigenvector.

The solution we get from the first eigenvalue and eigenvector is

$$\vec{x}_1(t) = e^{3\sqrt{3}it} \begin{pmatrix} 3 \\ -1+\sqrt{3}i \end{pmatrix}$$

We need to get rid of the complex numbers, so let's use Euler's formula. Then we will multiply the cosines and sines into the vector. We will then combine the terms with an "i" in them and split these terms off from those terms that don't contain an "i". Finally, we will factor out the "i".

$$\begin{aligned}
\vec{x}_1(t) &= (\cos(3\sqrt{3}t) + i\sin(3\sqrt{3}t))\begin{pmatrix} 3 \\ -1+\sqrt{3}i \end{pmatrix} \\
&= \begin{pmatrix} 3\cos(3\sqrt{3}t) + 3i\sin(3\sqrt{3}t) \\ -\cos(3\sqrt{3}t) - i\sin(3\sqrt{3}t) + \sqrt{3}i\cos(3\sqrt{3}t) - \sqrt{3}\sin(3\sqrt{3}t) \end{pmatrix} \\
&= \begin{pmatrix} 3\cos(3\sqrt{3}t) \\ -\cos(3\sqrt{3}t) - \sqrt{3}\sin(3\sqrt{3}t) \end{pmatrix} + i\begin{pmatrix} 3\sin(3\sqrt{3}t) \\ -\sin(3\sqrt{3}t) + \sqrt{3}\cos(3\sqrt{3}t) \end{pmatrix} \\
&= \vec{u}(t) + i\vec{v}(t)
\end{aligned}$$

It can be shown that $\vec{u}(t)$ and $\vec{v}(t)$ are two linearly independent solutions to the system. You can do that in your free time. This means that we can use them to form a general solution and they are both real solutions.

The general solution is

$$\vec{x}(t) = c_1\vec{u}(t) + c_2\vec{v}(t)$$

So we have:

$$\vec{x}(t) = c_1\begin{pmatrix} 3\cos(3\sqrt{3}t) \\ -\cos(3\sqrt{3}t) - \sqrt{3}\sin(3\sqrt{3}t) \end{pmatrix} + c_2\begin{pmatrix} 3\sin(3\sqrt{3}t) \\ -\sin(3\sqrt{3}t) + \sqrt{3}\cos(3\sqrt{3}t) \end{pmatrix}$$

Apply the initial conditions to this to find the constants:

$$\begin{pmatrix} 2 \\ -4 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} 3 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix}$$

Solving that systems of equations, we get

$$c_1 = \frac{2}{3},\ c_2 = -\frac{10}{3\sqrt{3}}$$

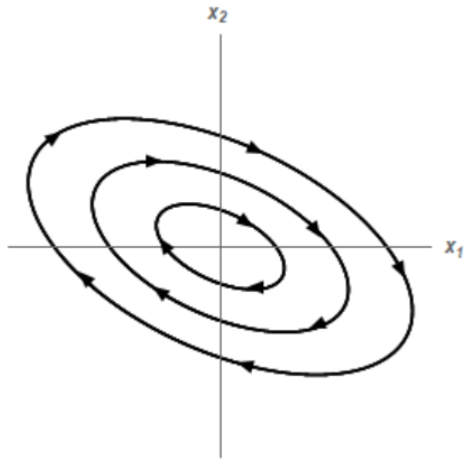
The actual solution is then:

$$\boldsymbol{\vec{x}(t) = \frac{2}{3}\begin{pmatrix} 3\cos(3\sqrt{3}t) \\ -\cos(3\sqrt{3}t) - \sqrt{3}\sin(3\sqrt{3}t) \end{pmatrix} - \frac{10}{3\sqrt{3}}\begin{pmatrix} 3\sin(3\sqrt{3}t) \\ -\sin(3\sqrt{3}t) + \sqrt{3}\cos(3\sqrt{3}t) \end{pmatrix}}$$

To sketch the phase portrait, let's pick a value of $\vec{x}(t)$ and plug it in:

$$\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies \vec{x}' = \begin{pmatrix} 3 & 9 \\ -4 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

Therefore, at the point $(1, 0)$, the trajectory will be pointing in a downwards direction. The only way that this can be is if the trajectories are traveling in a clockwise direction. Here is the sketch:



The equilibrium solution in this case is called a center and is stable.

Recitation 11.

11.25 Fourier Sine and Cosine Series

Like we discussed in class, both the Sine and Cosine Fourier series work well with odd and even functions respectively. So first, assume that we have some odd function $f(x)$. Because $f(x)$ is odd it makes some sense that we should be able to write a series representation for this in terms of sines only (since they are also odd functions).

The **Fourier sine series** of an odd function $f(x)$ on $-L \leq x \leq L$ is

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \\ B_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \quad n = 1,2,3,\ldots \\ &= \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \quad n = 1,2,3,\ldots \end{aligned}$$

where B_n are the **Fourier coefficients**.

Fact:

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,dx = \begin{cases} L & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,dx = \begin{cases} \frac{L}{2} & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Finding the Fourier sin series for a function that is not odd: Given a function $f(x)$, we define the **odd extension** of $f(x)$ to be the new function

$$g(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq L \\ -f(-x) & \text{if } -L \leq x \leq 0 \end{cases}$$

Note that this is an odd function, and because it is odd, we know that on $-L \leq x \leq L$, the Fourier sine series for $g(x)$ (and NOT $f(x)$ yet) is:

$$g(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \quad B_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \quad n = 1, 2, 3, \ldots$$

However, because we know that $g(x) = f(x)$ on $0 \leq x \leq L$, we can also see that as long as we are on $0 \leq x \leq L$, we have

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \quad B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \quad n = 1, 2, 3, \ldots$$

Basically, we wasted your time to read all that because this is the same formula for the coefficients.

Now for the Cos series, assume that $f(x)$ is an even function on $-L \leq x \leq L$.

The **Fourier cosine series** of an even function $f(x)$ on $-L \leq x \leq L$ is

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \\ A_n &= \begin{cases} \frac{1}{2L}\int_{-L}^{L} f(x)\,dx & n = 0 \\ \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx & n \neq 0 \end{cases} \\ &= \begin{cases} \frac{1}{L}\int_0^L f(x)\,dx & n = 0 \\ \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx & n \neq 0 \end{cases} \end{aligned}$$

I'll leave the derivations of both of these series up to the class notes. Let's do a couple of examples for now.

Example 11.58. Find the Fourier sine series for $f(x) = x$ on $-L \leq x \leq L$.
Solution. Compute the coefficients for $f(x) = x$:

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{2}{L}\left(\frac{L}{n^2\pi^2}\right)\left(L\sin\left(\frac{n\pi x}{L}\right) - n\pi x\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_0^L \qquad \text{left out integration by parts work} \\ &= \frac{2}{n^2\pi^2}(L\sin(n\pi) - n\pi L\cos(n\pi)) \end{aligned}$$

We know that since $n \in \mathbb{Z}$, $\sin(n\pi) = 0$ and that $\cos(n\pi) = (-1)^n$. We therefore have:

$$\begin{aligned} B_n &= \frac{2}{n^2\pi^2}(-n\pi L(-1)^n) \\ &= \frac{(-1)^{n+1}2L}{n\pi} \quad n = 1, 2, 3, \ldots \end{aligned}$$

The Fourier sine series is then:

$$\begin{aligned} x &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2L}{n\pi} \sin\left(\frac{n\pi x}{L}\right) \\ &= \boldsymbol{\frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{L}\right)} \end{aligned}$$

Example 11.59. Find the Fourier sine series for $f(x) = L - x$ on $-L \leq x \leq L$.

Solution. Compute coefficients:

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx \\ &= \frac{2}{L}\int_0^L (L-x)\sin\left(\frac{n\pi x}{L}\right)\, dx \\ &= \frac{2}{L}\left(-\frac{L}{n^2\pi^2}\right)\left[L\sin\left(\frac{n\pi x}{L} - n\pi(x-L)\cos\left(\frac{n\pi x}{L}\right)\right)\right]\Bigg|_0^L \\ &= \frac{2}{L}\left[\frac{L^2}{n^2\pi^2}(n\pi - \sin(n\pi))\right] \\ &= \frac{2L}{n\pi} \end{aligned}$$

Don't forget that n is an integer. The Fourier sine series is:

$$\boldsymbol{f(x) = \sum_{n=1}^{\infty} \frac{2L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)}$$

Find the Fourier sine series for $f(x) = 1 + x^2$ on $0 \leq x \leq L$.

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx \\ &= \frac{2}{L}\int_0^L (1+x^2)\sin\left(\frac{n\pi x}{L}\right) \\ &= \frac{2}{L}\left(\frac{2}{n^3\pi^3}\right)\left[(2L^2 - n^2\pi^2(1+x^2))\cos\left(\frac{n\pi x}{L} + 2Ln\pi x\sin\left(\frac{n\pi x}{L}\right)\right)\right]\Bigg|_0^L \qquad \text{int. by parts twice} \\ &= \frac{2}{n^3\pi^3}\left[(2L^2 - n^2\pi^2(1+L^2))(-1)^n - 2L^2 + n^2\pi^2\right] \end{aligned}$$

The Fourier sine series for this function is:

$$\boldsymbol{f(x) = \sum_{n=1}^{\infty} \frac{2}{n^3\pi^3} \left[(2L^2 - n^2\pi^2(1+L^2))(-1)^n - 2L^2 + n^2\pi^2\right] \sin\left(\frac{n\pi x}{L}\right)}$$

Example 11.60. Find the Fourier sine series for $f(x) = \begin{cases} \frac{L}{2} & \text{if } 0 \le x \le \frac{L}{2} \\ x - \frac{L}{2} & \text{if } \frac{L}{2} \le x \le L \end{cases}$ on $0 \le x \le L$.

Solution. Integral for coefficients:

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx \\ &= \frac{2}{L}\left[\int_0^{\frac{L}{2}} f(x)\sin\left(\frac{n\pi x}{2}\right) + \int_{\frac{L}{2}}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx\right] \end{aligned}$$

Note that we need to split the integral up because of the piecewise nature of the function. Let's do the two integrals separately:

$$\begin{aligned} \int_0^{\frac{L}{2}} \frac{L}{2}\sin\left(\frac{n\pi x}{L}\right)\, dx &= -\left(\frac{L}{2}\right)\left(\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\Bigg|_0^{\frac{L}{2}} \\ &= \frac{L^2}{2n\pi}\left(1 - \cos\left(\frac{n\pi}{2}\right)\right) \\ \int_{\frac{L}{2}}^{L}\left(x - \frac{L}{2}\right)\sin\left(\frac{n\pi x}{L}\right) &= \frac{L}{n^2\pi^2}\left[L\sin\left(\frac{n\pi x}{L}\right) - n\pi\left(x - \frac{L}{2}\right)\cos\left(\frac{n\pi x}{L}\right)\right]\Bigg|_{\frac{L}{2}}^{L} \\ &= \frac{L}{n^2\pi^2}\left[L\sin(n\pi) - \frac{n\pi L}{2}\cos(n\pi) - L\sin\left(\frac{n\pi}{2}\right)\right] \\ &= -\frac{L^2}{n^2\pi^2}\left[\frac{n\pi(-1)^n}{2} + \sin\left(\frac{n\pi}{2}\right)\right] \end{aligned}$$

Putting all of this together gives:

$$\begin{aligned} B_n &= \frac{2}{L}\left(\frac{L^2}{2n\pi}\right)\left[1 + (-1)^{n+1} - \cos\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi}\sin\left(\frac{2}{n\pi}\right)\right] \\ &= \frac{L}{n\pi}\left[1 + (-1)^{n+1} - \cos\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right] \end{aligned}$$

So the Fourier sine series is:

$$\boldsymbol{f(x) = \sum_{n=1}^{\infty} \frac{L}{n\pi}\left[1 + (-1)^{n+1} - \cos\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right]\sin\left(\frac{n\pi x}{2}\right)}$$

Example 11.61. Find the Fourier cosine series for $f(x) = x^2$ on $-L \le x \le L$.

Solution. We will need to do integration by parts twice but that will be left out.

$$\begin{aligned}
A_0 &= \frac{1}{L}\int_0^L f(x)\,dx \\
&= \frac{1}{L}\int_0^L x^2\,dx \\
&= \frac{1}{L}\left(\frac{L^3}{3}\right) \\
&= \frac{L^2}{3} \\
A_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \\
&= \frac{2}{L}\int_0^L x^2\cos\left(\frac{n\pi x}{L}\right)\,dx \\
&= \frac{2}{L}\left(\frac{L}{n^3\pi^3}\right)\left(2Ln\pi x\cos\left(\frac{n\pi x}{L}\right) + (n^2\pi^2x^2 - 2L^2)\sin\left(\frac{n\pi x}{L}\right)\right)\Bigg|_0^L \\
&= \frac{2}{n^3\pi^3}(2L^2n\pi\cos(n\pi) + (n^2\pi^2L^2 - 2L^2)\sin(n\pi)) \\
&= \frac{4L^2(-1)^n}{n^2\pi^2} \quad n = 1, 2, 3, \ldots
\end{aligned}$$

The Fourier cosine series is:

$$\begin{aligned}
x^2 &= \sum_{n=0}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) \\
&= A_0 + \sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{L}\right) \\
&= \boldsymbol{\frac{L^2}{3} + \sum_{n=1}^{\infty}\frac{4L^2(-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)}
\end{aligned}$$

Recitation 12.

12.26 Partial Differential Equations (PDE)

As mentioned earlier in this guide a partial differential equation is simply put a differential equation with partial derivatives in them. Deriving analytical solutions for this variety of equations are a little more involved and relies on a few assumptions as well. Other further classes will show how we can approximate solutions to partial differential equations empirically.

In this class we will discuss the method of **separation of variables** to solve linear and homogeneous

PDEs. This method relies on the assumption that a function of the form

$$x(x,t) = X(x)T(t)$$

will be a solution to the differential equation given that our boundary conditions (a lot like initial conditions) are also linear and homogeneous.

The process to solving these equations is as follow:

1. Assume a solution of the form $u(x,t) = X(x)T(t)$ where $X(x)$ and $T(t)$ are some function of x and t respectively and plug into the PDE.
2. Separate variables onto opposites sides of the equals sign.
3. Set both sides equal to some constant λ that we call the separation constant.
4. Use boundary conditions to solve the now separated ODEs.

This sounds confusing at a high level but let's look at a couple of examples.

Example 12.62. Do separation of variables on

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - u$$

with initial conditions $u(x,0) = X(x), u(0,t) = 0, -\frac{\partial u}{\partial x}(L,t) = u(L,t)$

Solution. The first step is to assume a generalized solution of the form $u(x,t) = X(x)T(t)$ and plug in to our PDE. We then get that

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - u \implies X(x)T'(t) = kX''(x)T(t) - X(x)T(t)$$

We now want to separate our variables from eachother.

$$\begin{aligned} X(x)T'(t) &= kX''(x)T(t) - X(x)T(t) \\ X(x)T'(t) &= T(t)(kX''(x) - X(x)) \\ \frac{T'(t)}{T(t)} &= k\frac{X''(x)}{X(x)} - 1 \end{aligned}$$

Now we set both sides equal to some constant λ. We know that both sides must equal some constant value because both sides of the equation are independent but equal. The only way this is true is if they are either the same exact expression - which we assume is not to be true else our original assumption of the form of the pde is redundant - or some constant value.

$$\frac{T'(t)}{T(t)} = k\frac{X''(x)}{X(x)} - 1 = \lambda$$

This then gives us two ODEs:

$$\frac{T'(t)}{T(t)} = \lambda, \quad k\frac{X''(x)}{X(x)} - 1 = \lambda$$

The initial boundary conditions, however, offer a few more clues about the final solution.

- Since we know that $u(x,0) = X(x)T(0) = X(x)$ we can conclude that $T(0) = 1$
- Since we know that $u(0,t) = X(0)T(t) = 0$ we can conclude that $X(0) = 0$
- Finally since we know that $-\frac{\partial u}{\partial x}(L,t) = -X'(L)T(t) = u(L,t) = X(L)T(t)$ we can conclude that $-X'(L) = X(L)$.

Each of these three conditions allow us to derive further condition for the ODEs we've derived above giving us a final step of having to apply our initial conditions to solve the ODEs.

Example 12.63. Use Separation of Variables on the following partial differential equation:

$$x^3 u_{tt} - t u_{xt} + x^3 u = 0$$

Solution. Plug into the DE:

$$x^3 X(x)T''(t) - tX'(x)T'(t) + x^3 X(x)T(t) = 0$$

Separate the variables and set equal to λ:

$$\frac{T''}{tT'} + \frac{T}{tT'} = \frac{X'}{x^3 X} = \lambda$$

This yields the two differential equations that we can now solve using other techniques discussed in the class

$$\boldsymbol{X' - \lambda x^3 X = 0}$$
$$\boldsymbol{T'' + T - \lambda t T' = 0}$$

Recitation 13.

13.27 Heat Equation

The heat equation is just a PDE used to model the heat and decay of certain objects. We've already discussed PDEs of a similar sort so I won't do any examples but in general, the Heat Equation is a PDE of the form

$$\begin{aligned} u_t &= k u_{xx} \\ u(x,0) &= f(x) \\ u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}$$

Where all the constants are dependent on the object that you are trying to model.

13.28 Wave Equation and Vibrating String

Consider a vertical string of length L that has been tightly stretched between two points at $x = 0$ and $x = L$. We will be solving the general problem for the 1-D wave equation to determine the displacement of a vibrating string:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \\ u(0,t) &= u(L,t) = 0 \end{aligned}$$

Start with separation of variables:

$$u(t,x) = X(x)T(t)$$

Plugging this into the two boundary conditions gives,

$$X(0) = 0 \quad X(L) = 0$$

Plugging the product solution into the differential equation, separating and introducing a separation constant gives

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ X(x)T''(t) &= c^2 T(t)X''(x) \\ \frac{1}{c^2 T}T'' &= \frac{1}{X}X'' = -\lambda \end{aligned}$$

We moved the c^2 to the left side for convenience and chose $-\lambda$ for the separation constant so the differential equation for X would match a known (and solved) case.

The two ODEs we get from separation of variables are then

$$\begin{aligned} T'' + c^2\lambda T &= 0 \\ X'' + \lambda X &= 0 \quad X(0) = 0 \quad X(L) = 0 \end{aligned}$$

From the boundary value problem in solving the heat equation, the eigenvalues and eigenfunctions for this problem are

$$\begin{aligned} \lambda_n &= \left(\frac{n\pi}{L}\right)^2 \\ X_n(x) &= \sin\left(\frac{n\pi x}{L}\right) \quad n = 1, 2, 3, \ldots \end{aligned}$$

The first ODE is now

$$T'' + \left(\frac{n\pi c}{L}\right)^2 T = 0$$

The general solution is

$$T(t) = c_1 \cos\left(\frac{n\pi ct}{L}\right) + c_2 \sin\left(\frac{n\pi ct}{L}\right)$$

The solution is

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi x}{L}\right)\right]$$

To solve for A_n and B_n, apply the initial conditions to get:

$$\begin{aligned} u(x,0) &= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \\ u_t(x,0) &= \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin\left(\frac{n\pi x}{L}\right) \end{aligned}$$

Example 13.64. Solve the problem

$$\begin{aligned} u_{tt} &= 81 u_{xx}, \quad 0 < x < 1, \quad t > 0 \\ u(0,t) &= u(1,t) = 0 \\ u(x,0) &= 3\sin(2\pi x), \quad u_t(x,0) = 2\sin(3\pi x) \end{aligned}$$

Solution. This looks a lot like the setup for the wave equation that we solved above. So we can go ahead and write:

$$u(x,t) = \sum_{n=1}^{\infty} [A_n \cos(9n\pi t) + B_n \sin(9n\pi t)] \sin(n\pi x)$$

To fill in the final steps we have that Solve for A_n:

$$\begin{aligned} u(x,0) &= \sum_{n=0}^{\infty} [A_n \cos(9n\pi \cdot 0) + B_n \sin(9n\pi \cdot 0)] \sin(n\pi x) \\ 3\sin(2\pi x) &= \sum_{n=0}^{\infty} A_n \sin(n\pi x) && n = 2 \\ A_2 &= 3 \end{aligned}$$

Solve for B_n:

$$\begin{aligned} u'(t) &= \sum_{n=1}^{\infty} [-9n\pi A_n \sin(9n\pi t) + 9n\pi B_n \cos(9n\pi t)] \sin(n\pi x) \\ 2\sin(3\pi x) &= \sum_{n=1}^{\infty} 9 B_n n\pi \sin(n\pi x) && n = 3 \\ &= 27 B_n \pi \sin(3\pi x) \\ B_3 &= \frac{2}{27\pi} \end{aligned}$$

So, the final solution is:

$$u(x,t) = 3\cos(18\pi t)\sin(2\pi x) + \frac{2}{27\pi}\sin(27\pi t)\sin(3\pi x)$$